Math 3353, Spring 2017
Due March 3

Homework 6 – Matrix Operations and Invertible Matrices

1. Given the matrices
   \[ A = \begin{bmatrix} 7 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -3 \\ -4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \]
   compute each of the following by hand:
   (a) \( A(BC) \) and \( (AB)C \)
   (b) \( (ABC)^T \) and \( A^T B^T C^T \)
   (c) \( AC \)

2. Determine whether each of the following statements is True or False. If any item is False, give a specific counterexample to show that the statement is not always True.
   (a) The matrix product \( AA^T \) is always well-defined, and is a square matrix.
   (b) If the matrix products \( BA \) and \( AB \) are well-defined, then \( AB \) and \( BA \) must be square matrices.
   (c) If the matrix products \( BA \) and \( AB \) are well-defined, then \( A \) and \( B \) must be square matrices.
   (d) The matrix power \( A^2 \) is always well-defined.

3. Write down the \( 3 \times 3 \) elementary matrices that perform the following steps:
   (a) \( E_1 \) adds row 1 to row 2, and then \( E_2 \) exchanges rows 2 and 3. What matrix \( M = E_2 E_1 \) does both operations at once?
   (b) \( \tilde{E}_1 \) exchanges rows 2 and 3, and then \( \tilde{E}_2 \) adds row 1 to row 3. What matrix \( \tilde{M} = \tilde{E}_2 \tilde{E}_1 \) does both operations at once?

   Explain in a sentence why \( M \) and \( \tilde{M} \) in the above parts are the same, but the individual matrices \( E_1, E_2, \tilde{E}_1 \) and \( \tilde{E}_2 \) are different.

4. If the product \( D = ABC \) of three square matrices is invertible, then \( A \) must be invertible (so are \( B \) and \( C \)). Find a formula for \( A^{-1} \) (i.e. \( A^{-1} = \cdots \)) that involves only the matrices \( B, B^{-1}, C, C^{-1}, D \) and/or \( D^{-1} \).
5. **MATLAB:** Matlab has a number of built-in matrices that arise frequently in scientific computing. Construct the $10 \times 10$ Hilbert matrix with the command $A = \text{hilb}(10)$, and create a vector $\mathbf{x} \in \mathbb{R}^{10}$ of all ones, $\mathbf{x} = \text{ones}(10,1)$. Use these to create a right-hand side vector $\mathbf{b} = A\mathbf{x}$, via the command $\mathbf{b} = A\mathbf{x}$.

You will solve the linear system $A\mathbf{x} = \mathbf{b}$ in two ways:

a. Solve using the “backslash” command (Matlab’s shortcut for performing Gaussian elimination) to get a solution vector, $y = A\backslash\mathbf{b}$.

b. Compute $A^{-1}$ using Matlab’s “inv” function, $\text{Ainv} = \text{inv}(A)$, and solve using the inverse matrix, $\mathbf{z} = A^{-1}\mathbf{b}$, via the command $\mathbf{z} = \text{Ainv*}\mathbf{b}$.

Compute the error vectors for both solutions, $\mathbf{x} - \mathbf{y}$ and $\mathbf{x} - \mathbf{z}$. Which result is more accurate (you’ll learn why in Math 3315/3316)?