1. Determine whether each of the following statements is True or False. If any item is False, give a specific counterexample (write down a relevant set of vectors with numbers in them) to show that the statement is not always true.

(a) If $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are in $\mathbb{R}^3$ and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

(b) If the vector equation $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0}$, where the vectors $\mathbf{v}_k$ are in $\mathbb{R}^3$, can only be solved with the constants $c_1 = c_2 = c_3 = 0$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

(c) The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \subset \mathbb{R}^3$ is linearly dependent.

(d) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \subset \mathbb{R}^4$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly dependent.

2. Let $A = \begin{bmatrix} -2 & 1 & 3 \\ 6 & 1 & -13 \\ -4 & 2 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, and consider the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. For each of the following, answer “yes” or “no” and explain your reasoning:

(a) Is $\mathbf{b}$ in the domain of $T$?

(b) Is $\mathbf{b}$ in the codomain of $T$?

(c) Is $\mathbf{b}$ in the range of $T$?

3. Consider the following transformation:

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_3 \\ 4x_1 - 2x_2 + x_3 \\ x_1 + x_3 \\ 2x_1 - x_3 \end{bmatrix}. $$

(a) What is the domain of $T$?

(b) What is the codomain of $T$?

(c) Show that $T$ is a linear transformation by finding the standard matrix of the transformation.
4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with $A$ its standard matrix. Complete the following statements, for each one the answer is either “row” or “column”:

(a) $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if every _________ of $A$ contains a pivot position.

(b) $T$ is one-to-one if and only if every _________ of $A$ contains a pivot position.

What do the above results tell you must be true about the shape of a matrix for the associated linear transformation to be both one-to-one and onto?

5. MATLAB: Consider the following matrices,

$$A = \begin{bmatrix}
1 & 1 & 0 & 0 & -1 & 5 \\
-5 & -2 & -5 & 9 & -16 & 9 \\
4 & 1 & 6 & -10 & 20 & -17 \\
-5 & -1 & -8 & 14 & -27 & 25 \\
2 & 1 & 1 & -2 & 3 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
5 & 1 & 4 & 2 & 11 \\
-3 & 0 & -3 & -1 & -8 \\
-4 & -1 & -3 & -1 & -9 \\
-2 & 1 & -1 & 1 & -5 \\
-7 & -4 & -6 & -5 & -14 \\
6 & 0 & 3 & 1 & 11
\end{bmatrix}.$$ 

Let $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = B\vec{x}$ be the linear transformations associated with each matrix.

(a) Is $T$ one-to-one?

(b) Does $T$ map $\mathbb{R}^6$ onto $\mathbb{R}^5$?

(c) Is $S$ one-to-one?

(d) Does $S$ map $\mathbb{R}^5$ onto $\mathbb{R}^6$?