Exam 2 Practice Solutions

Math 3353, Spring 2017

1. Compute
\[
\begin{vmatrix}
1 & -2 & 5 & 2 \\
0 & 0 & 3 & 0 \\
2 & -6 & -7 & 5 \\
5 & 0 & 4 & 0
\end{vmatrix}
\]
using a method of your choosing.

Solution: Since row two has 3 out of 4 zero entries, I’ll use cofactor expansion across the second row, followed by expansion across the third row, followed by the definition of the 2x2 determinant:
\[
\begin{vmatrix}
1 & -2 & 5 & 2 \\
0 & 0 & 3 & 0 \\
2 & -6 & -7 & 5 \\
5 & 0 & 4 & 0
\end{vmatrix} = -3 \begin{vmatrix}
1 & -2 & 2 \\
2 & -6 & 5 \\
5 & 0 & 0
\end{vmatrix} = (-3)(5) \begin{vmatrix}
-2 & 2 \\
-6 & 5 \\
0 & 0
\end{vmatrix} = (-3)(5)(2)
= -30.
\]

Alternately, you could have used row-reduction, ensuring to keep track of how row scaling and swapping modify the determinant:
\[
\begin{vmatrix}
1 & -2 & 5 & 2 \\
0 & 0 & 3 & 0 \\
2 & -6 & -7 & 5 \\
5 & 0 & 4 & 0
\end{vmatrix} = \begin{vmatrix}
1 & -2 & 5 & 2 \\
0 & 0 & 3 & 0 \\
0 & -2 & -17 & 1 \\
0 & 0 & 3 & 0
\end{vmatrix} = (-1)(-2)(3)(-5) = -30.
\]
2. If \[ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7, \] compute \[ \begin{vmatrix} 2d & 2e & 2f \\ 2a & 2b & 2c \\ 2g & 2h & 2i \end{vmatrix}. \]

Solution: The second matrix is formed by swapping rows 1 and 2 from the first matrix, which makes the determinant negative. The second matrix also scales each row by a factor of 2; since this scaling occurs on three rows, it changes the determinant by a factor of \(2^3\).

Hence the determinant of the second matrix equals \(-8)(7) = -56\).
3. Use Cramer’s rule to solve the linear system $A\vec{x} = \vec{b}$ for only the solution component $x_2$.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 0 \end{bmatrix}$$

**Solution:** To use Cramer’s rule to compute $x_2$, we compute the formula

$$x_2 = \frac{|A_2(\vec{b})|}{|A|}.$$  

Due to the large number of zeros in the matrix, I’ll compute these determinants by using cofactor expansion across row 1 at every step:

$$|A_2(\vec{b})| = \begin{vmatrix} 2 & 4 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 & 0 \\ 4 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 2 [3(4 - 1) - (8 - 0)] - 4 [(4 - 1) - (0 - 0)]$$

$$= 2 [9 - 8] - 4 [3]$$

$$= 2 - 12$$

$$= -10$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 2 [2(4 - 1) - (2 - 0)] - [(4 - 1) - (0 - 0)]$$

$$= 2 [6 - 2] - [3]$$

$$= 8 - 3$$

$$= 5.$$

**Hence**

$$x_2 = \frac{-10}{5} = -2.$$
4. Let $W$ be the set of vectors of the form
\[
\begin{bmatrix}
0 \\
2a + b \\
3b - a \\
4a - 7b
\end{bmatrix},
\]
where $a$ and $b \in \mathbb{R}$. If $W$ is a vector space find a set of vectors that spans $W$; otherwise prove that $W$ is not a vector space.

**Solution:** A set of vectors that spans $W$ is
\[
W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -7 \end{bmatrix} \right\} = \text{Span}\{ \vec{u}, \vec{v} \}.
\]

The problem is finished at this point.

However, for further explanation, let’s note that since $W$ can be written as a span of a set of vectors, then $W$ is automatically a vector space. As we proved in class,
\[
\vec{0} = 0\vec{u} + 0\vec{v} \in \text{Span}\{ \vec{u}, \vec{v} \}.
\]

Furthermore, if $\vec{x}$ and $\vec{y}$ are in $W$ then $\vec{x} = c_1\vec{u} + c_2\vec{v}$ and $\vec{y} = d_1\vec{u} + d_2\vec{v}$ for appropriate constants $c_1, c_2, d_1, d_2 \in \mathbb{R}$, so
\[
\begin{align*}
\quad a\vec{x} + b\vec{y} &= a (c_1\vec{u} + c_2\vec{v}) + b (d_1\vec{u} + d_2\vec{v}) \\
&= (ac_1 + bd_1)\vec{u} + (ac_2 + bd_2)\vec{v} \\
&= p\vec{u} + q\vec{v} \in \text{Span}\{ \vec{u}, \vec{v} \},
\end{align*}
\]
where $p = (ac_1 + bd_1) \in \mathbb{R}$ and $q = (ac_2 + bd_2) \in \mathbb{R}$, so $W$ is a subspace of $\mathbb{R}^4$, and hence it is a vector space.
5. Consider \( A = \begin{bmatrix} 2 & -5 & -2 & 6 & 1 \\ -2 & 5 & 0 & 1 & 0 \end{bmatrix} \). Find \( p \) such that \( \text{Nul}(A) \) is a subspace of \( \mathbb{R}^p \). Find \( q \) such that \( \text{Col}(A) \) is a subspace of \( \mathbb{R}^q \). Write two nonzero vectors, \( \vec{x} \in \text{Nul}(A) \) and \( \vec{y} \in \text{Col}(A) \).

**Solution:** Since \( A \in \mathbb{R}^{2 \times 5} \), then \( p = 5 \) and \( q = 2 \). To find a nonzero vector \( \vec{y} \in \text{Col}(A) \) we may select any linear combination of the columns of \( A \), e.g. the first column of \( A \):
\[
\vec{y} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.
\]

To find a nonzero vector \( \vec{x} \in \text{Nul}(A) \) we must first reduce \( A \) to echelon form,
\[
A = \begin{bmatrix} 2 & -5 & -2 & 6 & 1 \\ -2 & 5 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & -2 & 6 & 1 \\ 0 & 0 & -2 & 7 & 1 \end{bmatrix}.
\]
We have three free variables \( x_2, x_4 \) and \( x_5 \), so we may find \( \vec{x} \) by setting these free variables to any desired values (as long as at least one is nonzero), e.g. \( x_2 = 2, x_4 = x_5 = 0 \). Solving for the remaining two components of \( \vec{x} \) by using the two rows of the echelon form of \( A \), we have
\[
-2x_3 + 7x_4 + x_5 = 0 \quad \Rightarrow \quad 2x_3 = 0 \quad \Rightarrow \quad x_3 = 0,
\]
\[
2x_1 - 5x_2 - 2x_3 + 6x_4 + x_5 = 0 \quad \Rightarrow \quad 2x_1 - 10 = 0 \quad \Rightarrow \quad x_1 = 5.
\]

Hence a nonzero \( \vec{x} \in \text{Nul}(A) \) is \( \vec{x} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \).
6. Let \( A = \begin{bmatrix} 2 & 4 & -4 & 6 & 0 & 2 \\ -3 & -6 & 6 & -9 & 1 & 2 \\ -1 & -2 & 2 & -3 & 3 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \). Find bases for \( \text{Nul}(A) \) and \( \text{Col}(A) \). What are the dimensions of these two subspaces?

**Solution:** The easier half of the question is \( \text{Col}(A) \), where the basis consists of each column from \( A \) that contains a pivot when reduced to echelon form:

\[
\text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}.
\]

Since \( \text{Col}(A) \) has two basis vectors, it has dimension 2.

For \( \text{Nul}(A) \) we merely find the solution set to \( A\vec{x} = \vec{0} \) in parametric vector form. We have four free variables, \( x_2, x_3, x_4 \) and \( x_6 \). The remaining two equations require that

\[
x_5 + 5x_6 = 0 \quad \Rightarrow \quad x_5 = -5x_6 \\
x_1 + 2x_2 - 2x_3 + 3x_4 + x_6 = 0 \quad \Rightarrow \quad x_1 = -2x_2 + 2x_3 - 3x_4 - x_6.
\]

Putting these together, we have

\[
\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -5 \end{bmatrix} \right\}.
\]

Since \( \text{Nul}(A) \) has four basis vectors, it has dimension 4.
7. Suppose that $(\vec{v}, \alpha)$ is an eigenpair of an invertible matrix $A$ (hence $\vec{v}$ is also an eigenvector of $A^{-1}$). Also suppose that $(\vec{v}, \beta)$ is an eigenpair of a matrix $B$. Show that $\vec{v}$ is an eigenvector of the matrix $C = (3A^{-1} + B)$, and find its corresponding eigenvalue, $\lambda$.

**Solution:** To show that $\vec{v}$ is an eigenvector of $C$, we merely multiply $C\vec{v}$ and show that it equals $\lambda \vec{v}$. The resulting factor $\lambda$ will be the eigenvalue.

First, we note that since $A$ is invertible then $A^{-1}$ exists, and the eigenvalue $\alpha \neq 0$, so

\[
A\vec{v} = \alpha \vec{v} \iff A^{-1}A\vec{v} = A^{-1}\alpha \vec{v} \\
\iff \vec{v} = \alpha A^{-1}\vec{v} \\
\iff \frac{1}{\alpha} \vec{v} = \frac{\alpha}{\alpha} A^{-1}\vec{v} \\
\iff \frac{1}{\alpha} \vec{v} = A^{-1}\vec{v},
\]

so $\frac{1}{\alpha}$ is the eigenvalue of $A^{-1}$ corresponding to the eigenvector $\vec{v}$.

We now multiply:

\[
C\vec{v} = (3A^{-1} + B)\vec{v} = 3A^{-1}\vec{v} + B\vec{v} = \frac{3}{\alpha} \vec{v} + \beta \vec{v} = \left(\frac{3}{\alpha} + \beta\right) \vec{v} = \lambda \vec{v},
\]

where the eigenvalue is $\lambda = \frac{3}{\alpha} + \beta$. 
8. Assume that \( A = QBQ^{-1} \), where
\[
A = \begin{bmatrix}
-15 & 21 & -7 \\
-12 & 17 & -5 \\
-8 & 12 & -2
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
-1 & 0 & 0 \\
2 & 2 & 0 \\
-1 & 2 & -1
\end{bmatrix}.
\]
Find the eigenvalues of \( A \).

\textit{Solution:} There are two ways to do this problem, the very easy way and the harder way. The harder way would be to directly compute the characteristic polynomial \(|A - \lambda I|\) and factor the resulting cubic polynomial to find its roots.

The very easy way is to realize that the problem is saying that \( A \) and \( B \) are related by a similarity transformation, \( A = QBQ^{-1} \) or \( B = Q^{-1}AQ \), which indicates that the two matrices have the same eigenvalues (as we proved in class).

Furthermore, since \( B \) is lower-triangular, its eigenvalues are just the values on the diagonal, meaning that
\[
\lambda(B) = \{-1, 2, -1\}.
\]
Note: I listed -1 twice just to show its algebraic multiplicity; this is not required.

\textit{Hence the eigenvalues of} \( A \) \textit{are also} \( \lambda(A) = \{-1, 2, -1\} \).
9. The matrix \( A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \) has eigenvalue/eigenvector pairs \( \lambda_1 = 5, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \lambda_2 = -2, \vec{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \). Diagonalize \( A \) (i.e. write 3 matrices, \( P \), \( D \) and \( P^{-1} \) such that \( A = PDP^{-1} \)).

**Solution:** Since we are given all of the eigenvalues and eigenvectors of \( A \), this problem only requires that we organize this information and compute one 2x2 matrix inverse.

First, \( P \) contains the two eigenvectors and \( D \) contains the two eigenvalues on its diagonal. The order of these must match. I’ll put them in the order as stated,

\[
P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}.
\]

To finish the diagonalization, we must compute \( P^{-1} \). For a 2x2 matrix we have the simple analytical formula

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},
\]

so

\[
P^{-1} = \frac{1}{4 + 3} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4/7 & 3/7 \\ -1/7 & 1/7 \end{bmatrix}
\]

and our diagonalization is

\[
\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 4/7 & 3/7 \\ -1/7 & 1/7 \end{bmatrix}.
\]

Note: to check your work, you could multiply the three matrices on the right to see if they in fact equal \( A \).
10. The matrix $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$ has an eigenvector/eigenvalue pair $\lambda_1 = 3 - i$, $\vec{v}_1 = \begin{bmatrix} 5 \\ 2 + i \end{bmatrix}$.

(a) What is the other eigenpair?

**Solution:** Since $A \in \mathbb{R}^{2 \times 2}$ we may easily state that the other eigenpair consists of the complex conjugates of this eigenpair, i.e. $\lambda_2 = 3 + i$ and $\vec{v}_2 = \begin{bmatrix} 5 \\ 2 - i \end{bmatrix}$.

(b) Find an invertible matrix $P$ and a rotation matrix $C$ such that $A = PCP^{-1}$.

**Solution:** As we learned in class, these matrices may be constructed using a complex eigenpair of $A$ itself. We can use either eigenpair; here I’ll use the one that was provided:

$$P = \begin{bmatrix} \text{Re}(\vec{v}_1) & \text{Im}(\vec{v}_1) \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \text{Re}(\lambda_1) & \text{Im}(\lambda_1) \\ -\text{Im}(\lambda_1) & \text{Re}(\lambda_1) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}.$$