Practice Problems, Section 3.2

(12) (a) \( x_{n+1} = \frac{1}{3} \left( 2x_n - \frac{c}{x_n^2} \right) \)

This is a fixed-point iteration, with
\[ g(x) = \frac{1}{3} \left( 2x - \frac{c}{x^2} \right) \]

We analyze fixed-point convergence by computing \( |g'(x)| \). If this is less than 1 for all \( x \), then we can guarantee convergence.

\[ g'(x) = \frac{1}{3} \left( 2 + \frac{2c}{x^3} \right) \]

Consider \( c = 1 \) and \( x_0 = \frac{1}{10} \). Here, \( |g'(x_0)| = 0.667 > 1 \).

Hence, we cannot guarantee convergence.

[Yes, I know the book says "yes", but I believe that they are wrong, in that this theory cannot guarantee convergence.]

(b) \( x_{n+1} = \frac{1}{2} x_n + \frac{1}{x_n} \Rightarrow g(x) = \frac{1}{2} x + \frac{1}{x} \)

\[ g'(x) = \frac{1}{2} - \frac{1}{x^2} \]

Again, for \( x_0 < 1 \), \( |g'(x_0)| > 1 \), so we cannot guarantee convergence for all \( x_0 \).

If (a) did converge, how do we determine the answer? Since "convergence" means \( x_{\text{new}} = x_n \), let's replace the iteration with the equation
\[
\frac{1}{3} \left( 2x - \frac{c}{x^2} \right) \Rightarrow x - \frac{2}{3}x = -\frac{c}{3x^2} \Rightarrow \frac{x}{3} = -\frac{c}{3x^2} \\
\Rightarrow x = -\frac{c}{x^2} \Rightarrow x^3 = -c \Rightarrow x = \sqrt[3]{-c} = -c.\]
17) \[ f(x) = x^5 - x^3 + 3, \quad x_n = 1 \]
\[ \Rightarrow f'(x) = 5x^4 - 3x^2 \]
\[ \Rightarrow X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)} = 1 - \frac{1 - 1 + 3}{5 - 3} = 1 - \frac{3}{2} = \frac{-1}{2} \]

41) New algorithm, where one step is two Newton steps:
\[ \tilde{x} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x_{n+1} = \tilde{x} - \frac{f(x_n)}{f'(x_n)} \]

Order of convergence: We know that
\[ |\tilde{x} - x^*| \leq C |x_n - x^*|^2 \]
and
\[ |x_{n+1} - \tilde{x}| \leq C |\tilde{x} - x^*|^2 \]
Since both parts are full Newton steps.
Combining these, we have
\[ |x_{n+1} - x^*| \leq C |\tilde{x} - x^*|^2 \leq C (C |x_n - x^*|^2)^2 \]
\[ \Rightarrow 4\text{th-order convergence?} \]