Exam 1 (Practice)

Math 3353, Spring 2017

The following will be the instructions on your in-class exam (the number of questions may vary):

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).

- Justify your answers algebraically whenever possible to ensure full credit.

- No calculators or other electronic devices are permitted without the explicit consent of the instructor.

- This exam is open book and open notes.

- Circle or otherwise indicate your final answers.

- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.

- This test has 8 problems and is worth 160 points. It is your responsibility to make sure that you have all of the pages!

- Good luck!

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1. Consider the linear system

\[\begin{align*}
4x + by &= 7, \\
-2x + 5y &= g
\end{align*}\]

(a) For what value of \(b\) is the matrix singular?

(b) Using your value of \(b\), determine a value of \(g\) so that the system is consistent.

(c) For your values of \(b\) and \(g\) there are an infinite number of solutions \(x\) and \(y\) to the system. Give two such solutions.
2. Determine whether each of the following statements is True or False. If any item is False, explain your reasoning or give a counterexample.

(a) The equation $A\vec{x} = \vec{b}$ is referred to as a vector equation.

(b) A vector $\vec{b}$ is a linear combination of the columns of a matrix $A$ if and only if the equation $A\vec{x} = \vec{b}$ has a unique solution.

(c) If the columns of a matrix $A \in \mathbb{R}^{m \times n}$ span $\mathbb{R}^m$, then the equation $A\vec{x} = \vec{b}$ is consistent for any $\vec{b} \in \mathbb{R}^n$.

(d) The equation $A\vec{x} = \vec{b}$ is consistent if the matrix $A$ has a pivot position in every column.
3. Consider the following augmented matrix,

\[
\begin{bmatrix}
1 & 7 & 3 & 1 & -4 \\
0 & 0 & 1 & -1 & -2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 2 & 3
\end{bmatrix}.
\]

Write the solution set of the system in parameteric vector form.
4. Determine if \( \vec{b} \) is a linear combination of \( \vec{a}_1 \) and \( \vec{a}_2 \); if so, give the weights on \( \vec{a}_1 \) and \( \vec{a}_2 \) used to generate \( \vec{b} \),

\[
\vec{b} = \begin{bmatrix} -7 \\ -3 \\ 39 \end{bmatrix}, \quad \vec{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}.
\]
5. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the transformation that reflects each vector \( \vec{x} \) through the plane \( x_3 = 0 \), i.e. \( T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix} \). Prove that \( T \) is a linear transformation. Is this transformation one-to-one? Is this transformation onto?
6. If $\vec{u}$ and $\vec{v}$ are both column vectors in $\mathbb{R}^n$, then we may define the “inner products” $\vec{u}^T \vec{v}$ and $\vec{v}^T \vec{u}$, and the “outer products” $\vec{u} \vec{v}^T$ and $\vec{v} \vec{u}^T$.

(a) What are the dimensions of the inner products $\vec{u}^T \vec{v}$ and $\vec{v}^T \vec{u}$?

(b) What are the dimensions of the outer products $\vec{u} \vec{v}^T$ and $\vec{v} \vec{u}^T$?

(c) How are the inner products $\vec{u}^T \vec{v}$ and $\vec{v}^T \vec{u}$ related (i.e. write $\vec{u}^T \vec{v}$ in terms of $\vec{v}^T \vec{u}$)?

(d) How are the outer products $\vec{u} \vec{v}^T$ and $\vec{v} \vec{u}^T$ related?
7. Assume that $A \in \mathbb{R}^{n \times n}$ is invertible. Prove that $A^T$ is invertible, and that

$$(A^T)^{-1} = (A^{-1})^T.$$ 

Do not just quote this theorem from section 2.2, or the theorem from section 2.3; prove this directly using the definition of “inverse” and the properties of matrix arithmetic from section 2.1.
8. Suppose $A \in \mathbb{R}^{3 \times 3}$ has the factorization $A = PDP^{-1}$, where $P \in \mathbb{R}^{3 \times 3}$ is some invertible matrix, and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Find simple formulas for $A^2$, $A^3$ and $A^k$, where $k$ is a positive integer, using $P$, $P^{-1}$ and this matrix $D$. 